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VIBRATION OF SHALLOW CONICAL SHELLS WITH SHEAR FLEXIBILITY: A FIRST-ORDER THEORY

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Abstract—A formulation of shear deformation theory implemented numerically for the prediction of vibratory characteristics of shallow conical shell panels is presented. The derivation of thickness shear is assumed in a linear approximation. The Lamé parameter for the transverse shear strain component, which has previously been neglected, is considered. This consideration accounts for the replacement of a term in transverse strain distribution through the shell thickness which results in linear transverse shear strain distribution in contrast to the constant distribution hitherto known to researchers in this field. The energy integral, which incorporates the shear deformation and rotary inertia, is minimized to derive the governing eigen-matrix equation. A set of benchmark frequency solutions is presented for two exemplary conical shells: the cantilever and the fully clamped shells. Some selected mode shapes in terms of mid-surface contour plots and three-dimensional meshes are also illustrated.

1. INTRODUCTION

Conical shell panels have been used intensively in various engineering designs, particularly in aerospace, structural and marine engineering disciplines. The applications range from huge civil structures to small turbomachinery blades. To ensure a safe design, a detailed understanding of the vibratory characteristics of these structures must first be determined. In an effort to model the turbomachinery blades, the authors attempt to propose a first-order shear deformation theory together with an efficient *pb*-2 Ritz method to study the vibratory characteristics of the moderately thick shallow conical shells.

Numerous research has been done on free vibration of turbomachinery blades (Rao, 1977, 1980; Leissa *et al.*, 1982). Most of the works have been reviewed by Rao (1973), Leissa (1981) and Chang (1981). However, almost all the analyses on vibration of conical shells were based on Kirchhoff–Love's theory (Irie *et al.*, 1982, 1984; Teichmann, 1985; Srinivasan and Krishnan, 1987; Cheung *et al.*, 1989; Liew and Lim, 1994; Liew *et al.*, 1994, 1995; Lim and Liew, 1995) which ignored the transverse shear strain and rotary inertia. These analyses are, therefore, valid only for either a thin plate or a thin shell. Moreover, most of the references provide us with only the results for conical frustums. The free vibration of truncated conical shells of variable thickness using a transfer matrix approach and an extensive set of natural frequency data for truncated conical shells of uniform thickness under nine combinations of boundary conditions has been reported (Irie *et al.*, 1982, 1984).

Research on thin open conical shell panels, to the authors' knowledge, can only be found in a few references. Teichmann (1985) presented approximate solutions of fundamental frequencies and buckling loads of cylindrical and conical shell panels under initial stress. Srinivasan and Krishnan (1987) solved the free vibration frequencies of fully clamped conical shell panels based on Donnell's theory and using an integral equation approach. The free vibration frequencies of these type of shell panels were also investigated by Cheung *et al.* (1989) who employed a spline finite strip method.

Comparatively few publications are available for vibration of thick conical shells and all of them deal with conical frustums. Takahashi *et al.* (1982, 1985, 1986) studied the free

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vibration of thick truncated conical shells with variable thickness using various methods ranging from the Lagrangian approach, the classical stress–strain relations and the three-dimensional Hooke's law and a second-order shear deformation theory with components of displacements assumed in quadratic expressions of thickness. Srinivasan and Hosur (1989) solved the axisymmetric vibration of thick conical frustums of linearly varying thickness clamped at both ends with the consideration of shear deformation and rotary inertia. Sivadas and Ganesan (1992) analyzed the thick composite conical frustums using a higher-order shell theory with three-noded isoparametric axisymmetric finite elements. Furthermore, the accuracy of solutions using three different theories were compared: Love's first approximation shell theory; an improved theory with shear deformation and rotary inertia; and a shell theory with thickness normal strain and shear deformation.

None of the above references can be found for free vibration of shear deformable conical shell panels despite the practical engineering importance of these structures. Consequently, it is the key objective of the present paper to develop a shear deformation theory for free vibration analysis of moderately thick shallow conical shells.

An extremum energy approach based on the Ritz principle is applied here to examine the vibratory characteristics of a class of moderately thick shallow conical shell panels. These shear deformable shells have thickness less than 20% of their width in accordance with first-order shear deformation theory (Mindlin, 1951). The Lamé parameter for the transverse shear strain component, which has previously been neglected (Reddy, 1984; Kabir and Chaudhuri, 1991), is considered. This consideration accounts for the replacement of a term in transverse shear strain distributions through the shell thickness which results in linearly varying transverse shear strain distribution in contrast to the constant distribution. The actual transverse shear strain distribution is not a linear function but rather a non-linear function of thickness. The assumption of linear shear strain distribution, however, is compensated by introducing a shear correction factor for the resulting discrepancy.

In this analysis, the orthogonal displacement and rotation fields are approximated by their respective global shape functions. These shape functions are extremely flexible in accounting for various boundary conditions by fixing an appropriate power to the respective basic functions which constitute the shape functions. A governing eigen-matrix equation results. Comparatively little computational execution time and memory with respect to other domain discretization numerical methods is required because the present global method involves no mesh generation.

A set of comprehensive numerical dimensionless frequencies is presented. A convergence study is carried out to verify the downward convergence of eigenvalues. Comparison of results for thin conical panels is shown by assigning a small thickness ratio and the effects of various geometric parameters on the resonance characteristics are discussed. In addition, some selected vibration modes in mid-surface contour plots and three-dimensional meshes are included to illustrate the vibratory nature of conical shell panels. Apparently, no theoretical formulation for this type of moderately thick shell panels has been presented, therefore the present solutions may be regarded as benchmark numerical data for either design purposes or future reference.

2. STRAIN FIELDS AND EIGEN-MATRIX EQUATION

Consider a homogeneous, isotropic, moderately thick shallow conical shell panel with panel length a, reference width b_o, thickness h, cone length s, vertex angle θ_v and base subtended angle θ_o, as illustrated in Fig. 1. The cone base can be assumed to be elliptical with minor and major radii α_o and β_o since the shell panel is shallow. The radius of curvature in the chordwise direction $R_y(x, y)$ is a parameter varying in the x- and y-directions. The variation in the x-direction is linear. There is no curvature along the spanwise direction ($R_x = \infty$). The conical shell panels investigated here are the cantilever shell (CFFF) clamped at $x = 0$ and the fully clamped shell (CCCC).

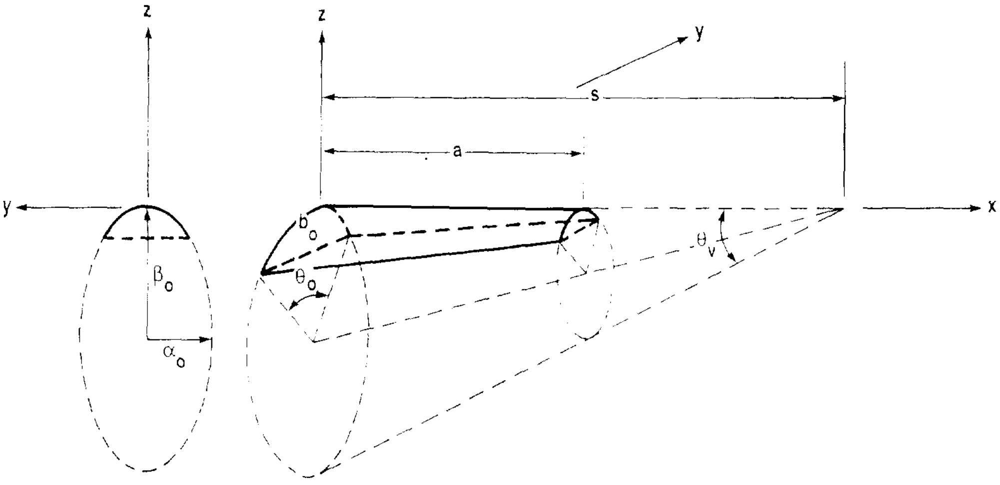

Fig. 1. Geometry of the shallow conical shell.

The geometry of a shallow conical shell with a trapezoidal planform is rather complex. From Fig. 1, the reference major radius and the major and minor radii α and β at any vertical cross-section are

$$\beta_o = s \tan\frac{\theta_v}{2} \tag{1a}$$

$$\beta = \tan\frac{\theta_v}{2}(s-x) \tag{1b}$$

$$\alpha = \frac{b\beta \tan(\theta_o/2)}{\sqrt{[4\beta^2 \tan^2(\theta_o/2) - b^2]}}, \tag{1c}$$

where

$$b = b_o\left[1-\left(\frac{x}{s}\right)\right] \tag{2a}$$

and

$$b_o = 2s \sin\frac{\theta_v}{2}\sqrt{\left(\frac{\tan^2\theta_o/2}{\cos^2\theta_v/2+\tan^2\theta_o/2}\right)}. \tag{2b}$$

The equation of an ellipse at any vertical cross-section is

$$\left(\frac{y}{\alpha}\right)^2 + \left(\frac{z+\beta}{\beta}\right)^2 = 1. \tag{3}$$

Taking the first and second derivatives of z with respect to y and making use of the definition $k = (d^2z/dy^2)/[1+(dz/dy)^2]^{3/2}$, the curvature at a point can be expressed as

$$k = -\frac{1}{\alpha^2\beta^2\left[\frac{1}{\beta^2}+\frac{y^2}{\alpha^2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\right]^{3/2}} \tag{4}$$

and the chordwise radius of curvature is

$$R_y(x,y) = \left|\frac{1}{k}\right| = \alpha^2\beta^2\left[\frac{1}{\beta^2}+\frac{y^2}{\alpha^2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\right]^{3/2}. \tag{5}$$

The orthogonal displacement components can be represented by first-order shear deformable functions of their corresponding displacements and rotations at the mid-surface in terms of the transverse coordinate as follows:

$$u = u_o + z\theta_1^u \tag{6a}$$

$$v = \left(1+\frac{z}{R_y(x,y)}\right)v_o + z\theta_1^v \tag{6b}$$

$$w = w_o, \tag{6c}$$

where (u, v, w) are the displacements of an arbitrary point along the x-, y-, z-directions and (u_o, v_o, w_o) are the displacements at the mid-surface. The present formulation assumes no transverse extensibility. The rotations of a point are denoted by θ_1^u and θ_1^v.

Using the above notations, the normal and shear strain fields are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{7a}$$

$$\varepsilon_y = \frac{1}{1+z/R_y(x,y)}\left(\frac{\partial v}{\partial y}+\frac{w}{R_y(x,y)}\right) \tag{7b}$$

$$\varepsilon_z = 0 \tag{7c}$$

$$\gamma_{yz} = \left[\frac{1}{1+z/R_y(x,y)}\right]\left(\frac{\partial w}{\partial y}-\frac{v}{R_y(x,y)}\right)+\frac{\partial v}{\partial z} \tag{7d}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \tag{7e}$$

and

$$\gamma_{xy} = \frac{1}{1+z/R_y(x,y)}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}, \tag{7f}$$

where the Lamé parameter $1/[1+z/R_y(x,y)]$ indicates the existence of shell curvature.

Substituting eqns (6a–c) into eqns (7a–f) yields a reduced strain field in functions of the displacements and rotations of the mid-surface:

$$\varepsilon_x = \frac{\partial u_o}{\partial x}+z\frac{\partial \theta_1^u}{\partial x} \tag{8a}$$

$$\varepsilon_y = \frac{\partial v_o}{\partial y}+\frac{w_o}{R_y(x,y)}+z\frac{\partial \theta_1^v}{\partial y}-\frac{z^2}{R_y(x,y)}\frac{\partial \theta_1^v}{\partial y} \tag{8b}$$

$$\varepsilon_z = 0 \tag{8c}$$

$$\gamma_{yz} = \frac{\partial w_o}{\partial y} + \theta_1^v - \left[\frac{z}{R_y(x,y)}\left(\frac{\partial w_o}{\partial y} + \theta_1^v\right)\right] \tag{8d}$$

$$\gamma_{xz} = \frac{\partial w_o}{\partial x} + \theta_1^u \tag{8e}$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\left[\frac{1}{R_y(x,y)}\left(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y}\right) + \frac{\partial \theta_1^u}{\partial y} + \frac{\partial \theta_1^v}{\partial x}\right] - \frac{z^2}{R_y(x,y)}\frac{\partial \theta_1^u}{\partial y}. \tag{8f}$$

Note that the neglect of the Lamé parameter for the transverse shear strain in square brackets in eqn (7d) by Reddy (1984) and Kabir and Chaudhuri (1991) results in the replacement of the term v/R_y in their papers by the term in square brackets in eqn (8d) here. Furthermore, these transverse shear strain components are affected by the presence of shell curvature. In the xz plane where there is no curvature, the transverse shear strain γ_{xz} in eqn (8e) is a constant through the thickness similar to a flat plate. However, in the yz plane, the shear strain γ_{yz} is a linear function of thickness as given in eqn (8d) in contrast to the constant distribution hitherto known to researchers in this field.

The strain and kinetic energies of a shallow conical shell are

$$\mathscr{U} = \frac{E}{2(1-\nu^2)}\iiint_V \left[\varepsilon_x^2 + 2\nu\varepsilon_x\varepsilon_y + \varepsilon_y^2 + \frac{1-\nu}{2}(\kappa^2\gamma_{yz}^2 + \kappa^2\gamma_{xz}^2 + \gamma_{xy}^2)\right]\mathrm{d}V \tag{9}$$

and

$$\mathscr{T} = \frac{\rho}{2}\iiint_V \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}V, \tag{10}$$

where E is the Young's modulus, ν Poisson's ratio, $\kappa^2 = 5/6$ is the shear correction factor, V the volume and ρ the mass density per unit volume. No normal stress is assumed in the above expressions.

From the assumption of small amplitude of free vibration, the translation and rotation fields can be approximated by sinusoidal functions as:

$$(u_o, v_o, w_o, \theta_1^u, \theta_1^v)(x,y,t) = (U_o, V_o, W_o, \Theta_1^u, \Theta_1^v)(x,y)\sin\omega t. \tag{11}$$

Substituting eqns (8a–f) and (11) into eqns (9) and (10), the maximum strain and kinetic energies in a vibratory cycle are, respectively,

$$\mathscr{U}_{\max} = \frac{D}{2}\iint_A \left\{\frac{12}{h^2}\left[\left(\frac{\partial U_o}{\partial x}\right)^2 + \left(\frac{\partial V_o}{\partial y}\right)^2 + \frac{2W_o}{R_y(x,y)}\frac{\partial V_o}{\partial y} + \left(\frac{W_o}{R_y(x,y)}\right)^2\right.\right.$$

$$\left. + 2\nu\frac{\partial U_o}{\partial x}\left(\frac{\partial V_o}{\partial y} + \frac{W_o}{R_y(x,y)}\right) + \frac{1-\nu}{2}\left(\frac{\partial U_o}{\partial y}\right)^2 + (1-\nu)\frac{\partial U_o}{\partial y}\frac{\partial V_o}{\partial x} + \frac{1-\nu}{2}\left(\frac{\partial V_o}{\partial x}\right)^2\right]$$

$$+ \frac{6\kappa^2(1-\nu)}{h^2}\left[\left(\frac{\partial W_o}{\partial y}\right)^2 + 2\frac{\partial W_o}{\partial y}\Theta_1^v + (\Theta_1^v)^2 + \left(\frac{\partial W_o}{\partial x}\right)^2 + 2\frac{\partial W_o}{\partial x}\Theta_1^u + (\Theta_1^u)^2\right]$$

$$+ \left(\frac{\partial \Theta_1^u}{\partial x}\right)^2 - \frac{2}{R_y(x,y)}\frac{\partial V_o}{\partial y}\frac{\partial \Theta_1^v}{\partial y} + \left(\frac{\partial \Theta_1^v}{\partial y}\right)^2 + 2\nu\left[\frac{\partial \Theta_1^u}{\partial x}\frac{\partial \Theta_1^v}{\partial y} - \frac{1}{R_y(x,y)}\frac{\partial U_o}{\partial x}\frac{\partial \Theta_1^v}{\partial y}\right]$$

$$+\frac{1-\nu}{2}\left[\frac{2}{R_y(x,y)}\frac{\partial V_o}{\partial x}\frac{\partial \Theta_1^v}{\partial x}-\frac{4}{R_y(x,y)}\frac{\partial U_o}{\partial y}\frac{\partial \Theta_1^u}{\partial y}-\frac{2}{R_y(x,y)}\frac{\partial U_o}{\partial y}\frac{\partial \Theta_1^v}{\partial x}+\left(\frac{\partial \Theta_1^u}{\partial y}\right)^2\right.$$

$$\left.\left.+2\frac{\partial \Theta_1^u}{\partial y}\frac{\partial \Theta_1^v}{\partial x}+\left(\frac{\partial \Theta_1^v}{\partial x}\right)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y \qquad (12)$$

and

$$\mathscr{T}_{\max}=\frac{\rho\omega^2}{2}\int\int_A\left\{h[U_o^2+V_o^2+W_o^2]+\frac{h^3}{12}\left[(\Theta_1^u)^2+\frac{2V_o\Theta_1^v}{R_y(x,y)}+\left(\frac{V_o}{R_y(x,y)}\right)^2+(\Theta_1^v)^2\right]\right\}\mathrm{d}x\,\mathrm{d}y, \qquad (13)$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity and A the projected planform area of the shell mid-surface.

For a generalized formulation independent of shell dimensions, a non-dimensional coordinate system is applied,

$$\xi=\frac{x}{a};\quad \eta=\frac{y}{b_o};\quad \zeta=\frac{z}{h}, \qquad (14)$$

where a, b and h are the length, width and thickness of the shell. Using this non-dimensional coordinate system, the displacement and rotation amplitude functions $U_o(\xi,\eta)$, $V_o(\xi,\eta)$, $W_o(\xi,\eta)$, $\Theta_1^u(\xi,\eta)$ and $\Theta_1^v(\xi,\eta)$ given in eq (11) can be represented by

$$U_o(\xi,\eta)=\sum_{i=1}^{m}c_i^u\phi_i^u(\xi,\eta) \qquad (15a)$$

$$V_o(\xi,\eta)=\sum_{i=1}^{m}c_i^v\phi_i^v(\xi,\eta) \qquad (15b)$$

$$W_o(\xi,\eta)=\sum_{i=1}^{m}c_i^w\phi_i^w(\xi,\eta) \qquad (15c)$$

$$\Theta_1^u(\xi,\eta)=\sum_{i=1}^{m}\bar{c}_i^u\Phi_i^u(\xi,\eta) \qquad (15d)$$

$$\Theta_1^v(\xi,\eta)=\sum_{i=1}^{m}\bar{c}_i^v\Phi_i^v(\xi,\eta) \qquad (15e)$$

in which $c_i^u, c_i^v, c_i^w, \bar{c}_i^u, \bar{c}_i^v$ are the unknown coefficients and $\phi_i^u, \phi_i^v, \phi_i^w, \Phi_i^u, \Phi_i^v$ are the corresponding shape functions.

The chordwise radius of curvature in the non-dimensional coordinate is

$$\frac{R_y(\xi,\eta)}{s}=\left(\frac{\alpha}{s}\right)^2\left(\frac{\beta}{s}\right)^2\left\{\left(\frac{s}{\beta}\right)^2+\eta^2\left(\frac{b_o}{s}\right)^2\left(\frac{s}{\alpha}\right)^2\left[\left(\frac{s}{\alpha}\right)^2-\left(\frac{s}{\beta}\right)^2\right]\right\}^{3/2}. \qquad (16)$$

Based on the principle of extremum energy, the following energy functional

$$\Pi=\mathscr{U}_{\max}-\mathscr{T}_{\max} \qquad (17)$$

is minimized with respect to the unknown coefficients

$$\frac{\partial \Pi}{\partial c_i^{\alpha}} = 0; \quad \alpha = u, v \text{ and } w \tag{18a}$$

and

$$\frac{\partial \Pi}{\partial \bar{c}_i^{\alpha}} = 0; \quad \alpha = u \text{ and } v \tag{18b}$$

which results in the governing eigenvalue equation,

$$(\mathbf{K} - \lambda^2 \mathbf{M})\{\mathbf{C}\} = \{\mathbf{0}\}. \tag{19}$$

The stiffness matrix **K**, mass matrix **M** and the coefficient vector **C** are:

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}^{uu} & \mathbf{k}^{uv} & \mathbf{k}^{uw} & \mathbf{k}^{u\bar{u}} & \mathbf{k}^{u\bar{v}} \\ & \mathbf{k}^{vv} & \mathbf{k}^{vw} & [0] & \mathbf{k}^{v\bar{v}} \\ & & \mathbf{k}^{ww} & \mathbf{k}^{w\bar{u}} & \mathbf{k}^{w\bar{v}} \\ & & & \mathbf{k}^{\bar{u}\bar{u}} & \mathbf{k}^{\bar{u}\bar{v}} \\ \text{sym} & & & & \mathbf{k}^{\bar{v}\bar{v}} \end{bmatrix} \tag{20a}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}^{uu} & [0] & [0] & [0] & [0] \\ & \mathbf{m}^{vv} & [0] & [0] & \mathbf{m}^{v\bar{v}} \\ & & \mathbf{m}^{ww} & [0] & [0] \\ & & & \mathbf{m}^{\bar{u}\bar{u}} & [0] \\ \text{sym} & & & & \mathbf{m}^{\bar{v}\bar{v}} \end{bmatrix} \tag{20b}$$

and

$$\{\mathbf{C}\} = \begin{Bmatrix} \{\mathbf{c}^u\} \\ \{\mathbf{c}^v\} \\ \{\mathbf{c}^w\} \\ a\{\bar{\mathbf{c}}^u\} \\ b\{\bar{\mathbf{c}}^v\} \end{Bmatrix}. \tag{20c}$$

The elements of the stiffness and mass matrices are:

$$k_{ij}^{uu} = \frac{12b^2}{h^2}\mathcal{I}^{1010}_{\phi_i^u\phi_j^u} + \frac{6a^2(1-\nu)}{h^2}\mathcal{I}^{0101}_{\phi_i^u\phi_j^u} \tag{21a}$$

$$k_{ij}^{uv} = \frac{12ab\nu}{h^2}\mathcal{I}^{1001}_{\phi_i^u\phi_j^v} + \frac{6ab(1-\nu)}{h^2}\mathcal{I}^{0110}_{\phi_i^u\phi_j^v} \tag{21b}$$

$$k_{ij}^{uw} = \frac{12ab^2\nu}{h^2\beta_a}\mathcal{I}^{1000}_{\phi_i^u\phi_j^w} \tag{21c}$$

$$k_{ij}^{u\theta_u} = -\frac{a(1-\nu)}{\beta_o}\mathcal{I}^{0101}_{\phi_i^u\Phi_j^u} \tag{21d}$$

$$k_{ij}^{u\theta_v} = -\frac{a\nu}{\beta_o}\mathcal{I}^{1001}_{\phi_i^u\Phi_j^v} - \frac{a(1-\nu)}{2\beta_o}\mathcal{I}^{0110}_{\phi_i^u\Phi_j^v} \tag{21e}$$

$$k_{ij}^{vv} = \frac{12a^2}{h^2}\mathcal{I}^{0101}_{\phi_i^v\phi_j^v} + \frac{6b^2(1-\nu)}{h^2}\mathcal{I}^{1010}_{\phi_i^v\phi_j^v} \tag{21f}$$

$$k_{ij}^{vw} = \frac{12a^2 b}{h^2\beta_o}\mathcal{I}^{0100}_{\phi_i^v\phi_j^w} \tag{21g}$$

$$k_{ij}^{v\theta_v} = \frac{b(1-\nu)}{2\beta_o}\mathcal{I}^{1010}_{\phi_i^v\Phi_j^v} - \frac{a^2}{b\beta_o}\mathcal{I}^{0101}_{\phi_i^v\Phi_j^v} \tag{21h}$$

$$k_{ij}^{ww} = \frac{6a^2\kappa^2(1-\nu)}{h^2}\mathcal{I}^{0101}_{\phi_i^w\phi_j^w} + \frac{6b^2\kappa^2(1-\nu)}{h^2}\mathcal{I}^{1010}_{\phi_i^w\phi_j^w} + \frac{12a^2b^2}{h^2\beta_o^2}\mathcal{I}^{0000}_{\phi_i^w\phi_j^w} \tag{21i}$$

$$k_{ij}^{w\theta_u} = \frac{6b^2\kappa^2(1-\nu)}{h^2}\mathcal{I}^{1000}_{\phi_i^w\Phi_j^u} \tag{21j}$$

$$k_{ij}^{w\theta_v} = \frac{6a^2\kappa^2(1-\nu)}{h^2}\mathcal{I}^{0100}_{\phi_i^w\Phi_j^v} \tag{21k}$$

$$k_{ij}^{\theta_u\theta_u} = \frac{6b^2\kappa^2(1-\nu)}{h^2}\mathcal{I}^{0000}_{\Phi_i^u\Phi_j^u} + \frac{b^2}{a^2}\mathcal{I}^{1010}_{\Phi_i^u\Phi_j^u} + \frac{(1-\nu)}{2}\mathcal{I}^{0101}_{\Phi_i^u\Phi_j^u} \tag{21l}$$

$$k_{ij}^{\theta_u\theta_v} = \nu\mathcal{I}^{1001}_{\Phi_i^u\Phi_j^v} + \frac{1-\nu}{2}\mathcal{I}^{0110}_{\Phi_i^u\Phi_j^v} \tag{21m}$$

$$k_{ij}^{\theta_v\theta_v} = \frac{6a^2\kappa^2(1-\nu)}{h^2}\mathcal{I}^{0000}_{\Phi_i^v\Phi_j^v} + \frac{a^2}{b^2}\mathcal{I}^{0101}_{\Phi_i^v\Phi_j^v} + \frac{1-\nu}{2}\mathcal{I}^{1010}_{\Phi_i^v\Phi_j^v} \tag{21n}$$

and

$$m_{ij}^{uu} = \mathcal{I}^{0000}_{\phi_i^u\phi_j^u}; \quad m_{ij}^{vv} = \mathcal{I}^{0000}_{\phi_i^v\phi_j^v} + \frac{h^2}{12\beta_o^2}\mathcal{I}^{0000}_{\phi_i^v\phi_j^v} \tag{22a,22b}$$

$$m_{ij}^{v\theta_v} = \frac{h^2}{12b\beta_o}\mathcal{I}^{0000}_{\phi_i^v\Phi_j^v}; \quad m_{ij}^{ww} = \mathcal{I}^{0000}_{\phi_i^w\phi_j^w} \tag{22c,22d}$$

$$m_{ij}^{\theta_u\theta_u} = \frac{h^2}{12a^2}\mathcal{I}^{0000}_{\Phi_i^u\Phi_j^u}; \quad m_{ij}^{\theta_v\theta_v} = \frac{h^2}{12b^2}\mathcal{I}^{0000}_{\Phi_i^v\Phi_j^v} \tag{22e,22f}$$

in which

$$\mathcal{I}^{defg}_{\varphi_i^\alpha\vartheta_j^\beta} = \iint_A \frac{\partial^{d+e}\varphi_i^\alpha(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\vartheta_j^\beta(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{23a}$$

$$\tilde{\mathcal{I}}^{defg}_{\varphi_i^\alpha\vartheta_j^\beta} = \iint_A f(\xi,\eta)\frac{\partial^{d+e}\varphi_i^\alpha(\xi,\eta)}{\partial\xi^d\partial\eta^e}\frac{\partial^{f+g}\vartheta_j^\beta(\xi,\eta)}{\partial\xi^f\partial\eta^g}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{23b}$$

where $\varphi, \vartheta = \phi, \Phi$; $\alpha, \beta = u, v, w$; $i, j = 1, 2, \ldots, m$, in which m is the total number of terms employed in the shape functions and

$$f(\xi, \eta) = \frac{\beta_o}{R_1(\xi, \eta)}, \tag{24}$$

where β_o is the reference major radius and $R_1(\xi, \eta)$ is the chordwise radius of curvature given in eqns (1a) and (16), respectively.

The governing eigenvalue eqn (19) can be solved readily by any standard numerical eigensolver where the eigenvalue or equivalently the non-dimensional frequency parameter is

$$\lambda = \frac{a\sqrt{[12(1-v^2)]}}{h}\lambda', \tag{25a}$$

where

$$\lambda' = \omega b_o \sqrt{\left(\frac{\rho}{E}\right)} \tag{25b}$$

is introduced because it is independent of the shell thickness h and thus enables the study of effects of h on λ'.

3. METHOD OF SOLUTION

The three translation and two rotation fields (U_o, V_o, W_o, Θ_1^u and Θ_1^v) can be approximated by respective global *pb*-2 shape functions which comprise the product of sets of orthogonally generated two-dimensional polynomials (*p*-2) and corresponding basic functions (*b*). The boundary geometric expressions which are raised to a basic power in accordance with the boundary constraints constitute the respective basic functions for each degree of freedom. Thus, the *pb*-2 shape functions satisfy the kinematic boundary conditions at the outset.

Due to symmetry of geometry and boundary constraints, the vibration modes of a shallow conical shell can be divided into two symmetry classes with respect to the xz-plane. For a CFFF or a CCCC shell panel, the vibration modes can be divided into symmetric (S) and antisymmetric (A) classes. Based on this criterion, the complete two-dimensional polynomial functions can be sub-divided into odd and even functions in terms of powers of ξ and η:

$$\sum_{i=1}^{m} f_i(\xi, \eta) = F_1(\xi, \eta_e) + F_2(\xi, \eta_o), \tag{26}$$

where

$$F_1(\xi, \eta_e) = \sum_{q=0}^{p} \sum_{i=0,2,4,\ldots}^{q} \xi^{q-i}\eta^i \tag{27a}$$

$$F_2(\xi, \eta_o) = \sum_{q=0}^{p} \sum_{i=1,3,5,\ldots}^{q} \xi^{q-i}\eta^i, \tag{27b}$$

Table 1. The two-dimensional functions for the symmetry classes

Boundary conditions (B.C.)	Symmetry class	U_o	V_o	W_o	Θ_1^u	Θ_1^v
CFFF	S	F_1	F_2	F_1	F_1	F_2
(or CCCC)	A	F_2	F_1	F_2	F_2	F_1

where p is the highest degree of polynomial in the functions. The two-dimensional functions of the translation and rotation fields for the symmetry classes are summarized in Table 1.

The pb-2 functions φ_i^α ($\varphi = \phi, \Phi$ and $\alpha = u, v, w$) for the respective degrees of freedom can be generated using the following orthogonal recursive procedure:

$$\varphi_i^\alpha(\xi,\eta) = f_i(\xi,\eta)\varphi_b^\alpha - \sum_{j=1}^{i-1} \Xi_{ij}^\alpha \varphi_j^\alpha, \tag{28}$$

where

$$\Xi_{ij}^\alpha = \frac{{}_1\Delta_{ij}^\alpha}{{}_2\Delta_j^\alpha} \tag{29a}$$

$${}_1\Delta_{ij}^\alpha = \iint_A f_i(\xi,\eta)\varphi_i^\alpha \varphi_j^\alpha \, d\xi \, d\eta \tag{29b}$$

$${}_2\Delta_j^\alpha = \iint_A (\varphi_j^\alpha)^2 \, d\xi \, d\eta$$

$$\varphi = \phi, \Phi \quad \text{and} \quad \alpha = u, v, w. \tag{29c}$$

The basic functions φ_b^α ($\varphi = \phi, \Phi$ and $\alpha = u, v, w$) are defined by the products of the equations of continuous piecewise boundary geometries of the shell planform each raised to an appropriate power that corresponds to the types of boundary constraints imposed on the shell. For a CFFF conical shell, the basic functions are:

$$\phi_b^u(\xi) = \phi_b^v(\xi) = \Phi_b^u(\xi) = \Phi_b^v(\xi) = \xi\,;\ \phi_b^w(\xi) = \xi^2 \tag{30}$$

and for a CCCC conical shell are:

$$\begin{aligned}\phi_b^u(\xi,\eta) &= \phi_b^v(\xi,\eta) = \Phi_b^u(\xi,\eta) = \Phi_b^v(\xi,\eta)\\ &= \xi\{\xi-1\}\left\{\eta-\frac{1}{2}\left(1-\frac{c}{b_o}\right)\left(\xi-\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{c}{b_o}\right)\right\}\\ &\quad\left\{\eta+\frac{1}{2}\left(1-\frac{c}{b_o}\right)\left(\xi-\frac{1}{2}\right)-\frac{1}{4}\left(1+\frac{c}{b_o}\right)\right\}\end{aligned} \tag{31a}$$

$$\begin{aligned}\phi_b^w(\xi,\eta) &= \xi^2\{\xi-1\}^2\left\{\eta-\frac{1}{2}\left(1-\frac{c}{b_o}\right)\left(\xi-\frac{1}{2}\right)\right.\\ &\quad\left.+\frac{1}{4}\left(1+\frac{c}{b_o}\right)\right\}^2\left\{\eta+\frac{1}{2}\left(1-\frac{c}{b_o}\right)\left(\xi-\frac{1}{2}\right)-\frac{1}{4}\left(1+\frac{c}{b_o}\right)\right\}^2,\end{aligned} \tag{31b}$$

where $c/b_o = 1 - a/s$ is the chord ratio.

Table 2. Convergence of frequency parameter $\lambda^* = \omega b_o\sqrt{(\rho/E)}$ for a shallow conical shell with $\nu = 0.3$, $\theta_v = 30$ and $\theta_o = 30$

B.C.	a/s	a/h	m	Symmetric modes S-1	S-2	S-3	S-4	Antisymmetric modes A-1	A-2	A-3	A-4
CFFF	0.1	30	40	0.12683	0.26604	0.43785	0.64690	0.11145	0.41622	0.58030	0.90359
			50	0.12681	0.26585	0.43764	0.64632	0.11137	0.41593	0.58011	0.90252
			60	0.12680	0.26579	0.43753	0.64606	0.11135	0.41578	0.58002	0.90204
			70	0.12678	0.26572	0.43745	0.64586	0.11132	0.41571	0.57994	0.90154
			80	0.12678	0.26569	0.43738	0.64569	0.11132	0.41563	0.57992	0.90122
		15	40	0.15648	0.50246	0.69043	1.1611	0.20673	0.76278	1.0684	1.1223
			50	0.15647	0.50218	0.69027	1.1599	0.20667	0.76241	1.0683	1.1220
			60	0.15645	0.50209	0.69020	1.1595	0.20664	0.76226	1.0682	1.1219
			70	0.15644	0.50204	0.69015	1.1593	0.20662	0.76218	1.0681	1.1219
			80	0.15644	0.50202	0.69012	1.1591	0.20661	0.76213	1.0681	1.1219
	0.2	30	40	0.045899	0.21460	0.46745	0.50435	0.095368	0.29829	0.39481	0.58724
			50	0.045894	0.21455	0.46737	0.50411	0.095338	0.29806	0.39478	0.58669
			60	0.045890	0.21452	0.46735	0.50401	0.095325	0.29797	0.39474	0.58652
			70	0.045888	0.21451	0.46734	0.50393	0.095317	0.29795	0.39473	0.58640
			80	0.045887	0.21450	0.46733	0.50390	0.095313	0.29792	0.39472	0.58634
		15	40	0.063333	0.33166	0.83192	0.91135	0.18294	0.39459	0.56298	1.0832
			50	0.063327	0.33160	0.83170	0.91120	0.18291	0.39456	0.56282	1.0827
			60	0.063320	0.33156	0.83160	0.91108	0.18289	0.39452	0.56275	1.0825
			70	0.063318	0.33155	0.83154	0.91103	0.18288	0.39450	0.56271	1.0823
			80	0.063316	0.33153	0.83149	0.91100	0.18287	0.39449	0.56268	1.0823
CCCC	0.1	30	40	0.64931	1.0704	1.1538	1.6262	0.73563	1.2503	1.7244	2.0018
			50	0.64930	1.0703	1.1536	1.6259	0.73557	1.2501	1.7240	2.0013
			60	0.64929	1.0703	1.1536	1.6256	0.73555	1.2501	1.7240	2.0010
			70	0.64929	1.0702	1.1535	1.6254	0.73554	1.2500	1.7238	2.0009
			80	0.64929	1.0702	1.1535	1.6253	0.73552	1.2500	1.7238	2.0007
		15	40	0.93463	1.8319	2.1251	2.9523	1.3244	2.2397	3.1583	3.5622
			50	0.93453	1.8314	2.1241	2.9508	1.3240	2.2388	3.1564	3.5596
			60	0.93447	1.8314	2.1238	2.9491	1.3239	2.2386	3.1561	3.5580
			70	0.93447	1.8311	2.1234	2.9482	1.3238	2.2381	3.1552	3.5578
			80	0.93442	1.8310	2.1232	2.9476	1.3237	2.2379	3.1550	3.5565
	0.2	30	40	0.70342	0.87088	1.1786	1.6291	1.1136	1.3571	1.6940	2.1453
			50	0.70341	0.87081	1.1783	1.6286	1.1134	1.3567	1.6932	2.1440
			60	0.70340	0.87080	1.1781	1.6285	1.1134	1.3565	1.6928	2.1437
			70	0.70340	0.87076	1.1781	1.6282	1.1133	1.3564	1.6927	2.1431
			80	0.70339	0.87075	1.1780	1.6281	1.1132	1.3562	1.6923	2.1428
		15	40	1.0205	1.3687	1.9524	2.7277	1.9568	2.3368	2.8668	3.5661
			50	1.0203	1.3681	1.9510	2.7252	1.9557	2.3347	2.8636	3.5611
			60	1.0202	1.3680	1.9501	2.7250	1.9555	2.3338	2.8619	3.5603
			70	1.0201	1.3677	1.9500	2.7235	1.9549	2.3332	2.8613	3.5577
			80	1.0201	1.3676	1.9493	2.7231	1.9547	2.3324	2.8597	3.5564

4. RESULTS AND DISCUSSION

4.1. *Convergence and comparison studies*

A convergence study is carried out to examine the numerical aspects of the present formulation. The convergence of eigenvalues for both CFFF and CCCC conical shells is shown in Table 2. Since it is impossible to undertake an all-encompassing survey of the convergence of eigenvalues for every case to be studied, only a few exemplary shell configurations are taken here for each of the CFFF and CCCC shells. The length ratios a/s investigated are 0.1 and 0.2 while the thickness ratios h/a are 1/30 ($h/b_o = 0.024$ for $a/s = 0.1$ and $h/b_o = 0.048$ for $a/s = 0.2$) and 1/15 ($h/b_o = 0.048$ for $a/s = 0.1$ and $h/b_o = 0.096$ for $a/s = 0.2$). The eigenvalues are presented in two symmetry classes in accordance with eqns (27a, b). The implication of the convergence is twofold: (1) the downward convergence of eigenvalues is monotonic which implies the overestimation of structural stiffness and thus vibration frequency by using the Ritz procedure; and (2) a total of 80 terms for each of the S and A symmetry classes is required in order to achieve a satisfactory numerical convergence of eigenvalues. Unless stated otherwise, all the subsequent results are calculated using $m = 80$ terms.

Table 3. Comparison of frequency parameter $\lambda = \omega a b_o \sqrt{(\rho h/D)}$ for a thin shallow conical shell with $\nu = 0.3$, $a/h = 200$, $a/s = 0.2$, $\theta_v = 15°$ and $\theta_o = 30°$

B.C.	Reference	Mode sequence number							
		1	2	3	4	5	6	7	8
CFFF	Lim and Liew (1995)	6.1727	9.0708	27.299	29.758	50.669	65.171	74.499	80.201
	Present	6.0586	9.0115	27.261	29.631	50.580	65.034	74.507	80.019
CCCC	Lim and Liew (1995)	221.95	254.84	288.49	298.36	318.16	326.43	335.10	351.26
	Present	221.68	254.40	287.87	297.89	317.71	325.59	334.64	350.81

Table 4. Comparison of frequency parameter $\lambda = \omega b_o \sqrt{(\rho/E)}$ for a moderately thick shallow conical shallow with $\nu = 0.3$, $a/h = 20.0$, $a/s = 0.1$ and $\theta_o = 30°$

B.C.	θ_v (deg)	Reference	Mode sequence number							
			S-1	S-2	S-3	S-4	A-1	A-2	A-3	A-4
CFFF	30	Reddy†	0.13261	0.38007	0.56269	0.90006	0.15593	0.58894	0.85038	1.0175
		Present	0.14120	0.38437	0.56362	0.90242	0.15876	0.59009	0.85576	1.0681
	60	Reddy†	0.31461	0.48562	0.87535	1.1281	0.28024	0.57408	1.1123	1.3281
		Present	0.31631	0.48972	0.87896	1.1285	0.28245	0.57762	1.1130	1.3319
CCCC	30	Reddy†	0.78488	1.4572	1.6522	2.3199	1.0304	1.7608	2.4781	2.8286
		Present	0.78652	1.4580	1.6572	2.3237	1.0348	1.7633	2.4838	2.8301
	60	Reddy†	1.2239	1.4727	2.2842	2.7800	1.2753	1.8105	2.8645	2.9124
		Present	1.2244	1.4749	2.2877	2.7801	1.2767	1.8134	2.8675	2.9137

†Independent calculations using the strain field of Reddy (1984) with shear correction factor 5/6.

A comparison study of vibration frequencies is provided in Table 3. The frequencies for a thin conical shell were obtained by Lim and Liew (1995) based on the Kirchhoff–Love hypothesis. Excellent agreement of the results has been achieved, indicating that the present formulation can simulate the vibration of a thin conical shell well by imposing a relatively small thickness ratio (in this case $h/a = 1/200$).

The effect of omitting the Lamé parameter in the transverse shear distribution in eqn (7d) is presented in Table 4. Reddy's first-order solutions in this table are obtained using the displacement field in eqns (6a–c) but replacing the strain field by the expressions presented by Reddy (1984):

$$\varepsilon_x = \frac{\partial u_o}{\partial x} + z\frac{\partial \theta_1^u}{\partial x} \tag{32a}$$

$$\varepsilon_y = \frac{\partial v_o}{\partial y} + \frac{w_o}{R_y(\xi,\eta)} + z\frac{\partial \theta_1^v}{\partial y} \tag{32b}$$

$$\varepsilon_z = 0 \tag{32c}$$

$$\gamma_{yz} = \frac{\partial w_o}{\partial y} + \theta_1^v - \frac{v_o}{R_y(\xi,\eta)} \tag{32d}$$

$$\gamma_{xz} = \frac{\partial w_o}{\partial x} + \theta_1^u \tag{32e}$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\left[\left(\frac{\partial \theta_1^u}{\partial y} + \frac{\partial \theta_1^v}{\partial x}\right) + \frac{1}{2R_y(\xi,\eta)}\left(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y}\right)\right] \tag{32f}$$

Table 5. Frequency parameter $\lambda' = \omega b_o \sqrt{(\rho/E)}$ for a CFFF shallow conical shell with $\nu = 0.3$ and $\theta_o = 15°$

			Mode sequence number							
θ_r			Symmetric modes				Antisymmetric modes			
(deg)	a/s	a/h	S-1	S-2	S-3	S-4	A-1	A-2	A-3	A-4
15	0.10	1000	0.0043469	0.018197	0.024155	0.026570	0.0029682	0.010505	0.022318	0.036697
		60	0.0074905	0.043737	0.11768	0.22258	0.038510	0.11251	0.11688	0.21010
		30	0.012898	0.077252	0.21193	0.40715	0.075059	0.11251	0.22702	0.40555
	0.15	1000	0.0020430	0.010668	0.024349	0.034701	0.0026122	0.0080521	0.015094	0.024089
		60	0.0042672	0.026781	0.073844	0.14302	0.038566	0.052488	0.11229	0.19389
		30	0.0077375	0.049895	0.13813	0.26732	0.051820	0.074460	0.21459	0.26138
	0.20	1000	0.0011977	0.0062014	0.015754	0.027852	0.0025746	0.0074537	0.013107	0.019738
		60	0.0018644	0.019109	0.053444	0.10430	0.029815	0.039623	0.11126	0.15663
		30	0.0030093	0.036452	0.10202	0.19805	0.029059	0.074434	0.15518	0.20965
30	0.10	1000	0.010646	0.020046	0.023298	0.039020	0.0065088	0.023268	0.033208	0.038939
		60	0.022229	0.10924	0.20292	0.25688	0.043731	0.14359	0.29119	0.39085
		30	0.031122	0.17091	0.39615	0.45081	0.085629	0.27870	0.39087	0.56000
	0.15	1000	0.0084050	0.018150	0.021954	0.030269	0.0040810	0.015750	0.031955	0.047585
		60	0.012176	0.065823	0.16838	0.30128	0.042285	0.12920	0.19815	0.24071
		30	0.019043	0.10930	0.29355	0.55392	0.082841	0.19811	0.25186	0.46551
	0.20	1000	0.0051395	0.019843	0.027961	0.028394	0.0033954	0.011460	0.023756	0.038666
		60	0.0082017	0.045586	0.12102	0.22827	0.042634	0.11836	0.12445	0.22210
		30	0.013699	0.079715	0.21672	0.41551	0.082962	0.11835	0.24143	0.42810
45	0.10	1000	0.010853	0.030009	0.042479	0.050205	0.012844	0.025652	0.031335	0.053107
		60	0.047913	0.14626	0.18908	0.30969	0.050401	0.18070	0.35255	0.40122
		30	0.059196	0.26523	0.30911	0.57908	0.096384	0.34038	0.69307	0.75769
	0.15	1000	0.011813	0.022094	0.027029	0.042043	0.0078498	0.025313	0.035655	0.041419
		60	0.025665	0.11902	0.21427	0.27072	0.046416	0.14999	0.30288	0.41531
		30	0.034365	0.18098	0.41718	0.46862	0.090600	0.29005	0.41534	0.58037
	0.20	1000	0.011992	0.018505	0.023540	0.034696	0.0054658	0.020590	0.036837	0.048026
		60	0.016618	0.083171	0.20325	0.29050	0.045739	0.13917	0.25941	0.26383
		30	0.024003	0.13148	0.34561	0.56871	0.089550	0.25960	0.27081	0.50853
60	0.10	1000	0.015794	0.033599	0.041171	0.068286	0.015657	0.037126	0.043732	0.051316
		60	0.083037	0.13869	0.25344	0.36240	0.061596	0.22992	0.27980	0.46631
		30	0.10185	0.25205	0.40002	0.62415	0.10991	0.41801	0.54627	0.89861
	0.15	1000	0.012197	0.031494	0.049348	0.051595	0.014289	0.027221	0.034074	0.054037
		60	0.049080	0.16057	0.19575	0.31823	0.052119	0.17945	0.38228	0.41025
		30	0.059013	0.26373	0.33972	0.59235	0.099309	0.33681	0.69160	0.72134
	0.20	1000	0.013218	0.025341	0.033389	0.046345	0.010093	0.027831	0.038456	0.045349
		60	0.031530	0.13524	0.22498	0.29169	0.049723	0.15840	0.31920	0.45129
		30	0.039860	0.19757	0.43694	0.49539	0.096461	0.30416	0.45135	0.60750

in which $R_y(\xi, \eta)$ is the variable curvature function. It is observed that the consideration of Lamé parameters for the transverse shear strains, which represent the presence of shell curvature, has significant effect on the vibration frequencies of a moderately thick shallow conical shell. The greatest discrepancy is about 6%, which corresponds to the S-1 mode of a CFFF shell with $\theta_r = 30°$.

4.2. *Numerical examples*

A set of results is presented in Tables 5–8. The effects of various shell geometric parameters, h/a, a/s, θ_r and θ_o, are examined. In order to study the influence of h/a on the frequency, the non-dimensional frequency parameter λ should be independent of thickness h and length a. Therefore, the results are presented in terms of λ' given in eqn (25b) instead of λ in eqn (25a). Tables 5 and 6 show λ' for a CFFF conical shell with $\theta_o = 15°$ and $30°$, respectively. The corresponding results for a CCCC conical shell are shown in Tables 7 and 8. For all calculations, the Poisson's ratio ν and shear correction factor used were 0.3 and 5/6, respectively.

It is well known that the first-order shear deformation theory provides relatively accurate solutions so long as the thickness is less than 20% of the shortest dimension. It is therefore necessary for the present calculation to be restricted by the above condition. The thickness ratio h/a provided in the tables may not reflect this requirement since the shortest dimension may either be the length of the panel a or the width b_o. The thickness to width ratio h/b_o can be determined using eqn (2b). The largest thickness ratio examined

Table 6. Frequency parameter $\lambda' = \omega b_o \sqrt{(\rho/E)}$ for a CFFF shallow conical shell with $\nu = 0.3$ and $\theta_o = 30°$

			Mode sequence number							
			Symmetric modes				Antisymmetric modes			
θ_r (deg)	a/s	a/h	S-1	S-2	S-3	S-4	A-1	A-2	A-3	A-4
15	0.10	1000	0.012122	0.028026	0.038164	0.050101	0.010539	0.028871	0.035803	0.049184
		30	0.040518	0.20333	0.40911	0.48691	0.085821	0.27936	0.37464	0.55979
		15	0.058148	0.31965	0.77256	0.84271	0.16520	0.37444	0.52899	1.0353
	0.15	1000	0.013449	0.020953	0.025267	0.038369	0.0058862	0.023883	0.041211	0.053056
		30	0.021662	0.12232	0.31591	0.57558	0.082354	0.18897	0.25093	0.46282
		15	0.033789	0.20378	0.54061	0.73957	0.15518	0.18939	0.47073	0.73485
	0.20	1000	0.0089683	0.023466	0.030207	0.033994	0.0043458	0.016420	0.034729	0.052047
		30	0.011356	0.084440	0.22668	0.42748	0.081611	0.11264	0.23990	0.42484
		15	0.015525	0.14705	0.39762	0.56094	0.10761	0.15488	0.43694	0.49046
30	0.10	1000	0.019491	0.039528	0.048030	0.077367	0.018541	0.044039	0.056396	0.058067
		30	0.12678	0.26569	0.43738	0.64569	0.11132	0.41563	0.57992	0.90122
		15	0.15644	0.50202	0.69012	1.1591	0.20661	0.76213	1.0681	1.1219
	0.15	1000	0.014570	0.037122	0.054330	0.061353	0.017791	0.032210	0.040016	0.062062
		30	0.070936	0.29029	0.36360	0.59829	0.098668	0.33072	0.61162	0.69717
		15	0.091811	0.44867	0.68869	1.0873	0.18892	0.61071	0.62150	1.2838
	0.20	1000	0.014929	0.030976	0.044074	0.053471	0.013208	0.031851	0.042697	0.052668
		30	0.045887	0.21450	0.46733	0.50390	0.095313	0.29792	0.39472	0.58634
		15	0.063316	0.33153	0.83149	0.91100	0.18287	0.39449	0.56268	1.0823
45	0.10	1000	0.026924	0.054792	0.064636	0.074757	0.027370	0.052176	0.063309	0.077234
		30	0.19610	0.31079	0.60862	0.74717	0.15653	0.43219	0.61141	0.95184
		15	0.27901	0.47550	1.0266	1.3386	0.25910	0.83312	1.0920	1.7740
	0.15	1000	0.023980	0.042593	0.054499	0.082275	0.022476	0.047853	0.061386	0.067900
		30	0.14683	0.28313	0.47177	0.68026	0.12085	0.43723	0.61552	0.92858
		15	0.17669	0.53180	0.73097	1.2049	0.22046	0.79383	1.1267	1.1874
	0.20	1000	0.019611	0.041942	0.055799	0.070926	0.022504	0.038749	0.049937	0.070451
		30	0.099586	0.32475	0.39388	0.63784	0.11043	0.37073	0.76854	0.78504
		15	0.12147	0.52983	0.68857	1.1671	0.20847	0.68611	0.77004	1.4404
60	0.10	1000	0.037264	0.067564	0.074035	0.088649	0.037256	0.065149	0.079578	0.093368
		30	0.23191	0.45322	0.59646	0.82446	0.22750	0.40723	0.79457	0.91357
		15	0.38690	0.54627	1.1559	1.4169	0.33949	0.75158	1.4253	1.7429
	0.15	1000	0.031796	0.058413	0.069367	0.079728	0.032579	0.053678	0.068773	0.089979
		30	0.21155	0.32537	0.61009	0.81743	0.16123	0.47720	0.60959	0.97270
		15	0.28933	0.50776	0.99636	1.3528	0.26378	0.91008	1.0808	1.7020
	0.20	1000	0.030481	0.047377	0.063189	0.089052	0.028387	0.052613	0.068233	0.082967
		30	0.17793	0.30381	0.52094	0.73215	0.13443	0.46770	0.65098	0.96210
		15	0.20947	0.56337	0.79319	1.2607	0.23807	0.83582	1.2079	1.2500

corresponds to $\theta_o = 30°$, $\theta_r = 15°$, $a/s = 0.20$, $h/a = 1/15$ and $h/b_o = 0.196$ (a moderately thick shell) in Tables 5 and 7 and the smallest corresponds to $\theta_o = 30°$, $\theta_r = 15°$, $a/s = 0.10$, $h/a = 1/1000$ and $h/b_o = 0.00033$ (a thin shell) in Tables 5 and 7. Thus, the present results cover a wide range of shell configurations ranging from thin to moderately thick shallow shells.

The effect of shell thickness is obvious. An increase in h/a (and an equivalent decline in a/h) results in higher shell bending stiffness and a higher vibration frequency λ'. These are the transverse dominant vibration modes. However, there are a few cases where λ' is insignificantly affected by an increase in h/a. For example, $\lambda' = 0.11251$ for the A-2 mode with $\theta_o = 15°$, $\theta_r = 15°$ and $a/s = 0.10$ invariable when h/a increases from 1/60 to 1/30 in Table 5. Similarly, λ' for the A-2 mode with $\theta_o = 15°$, $\theta_r = 30°$, $a/s = 0.20$ changes from 0.11836 to 0.11835 when h/a varies from 1/60 to 1/30 in Table 5. This does not contradict the statement above, where a thicker shell has higher bending stiffness, because these two vibration modes are effectively in-plane dominant, as we shall see later in the contour and three-dimensional vibration mode shapes.

The aspect ratio a/b_o of the conical shell is characterized by a/s and b_o/s. From eqn (2b), it can be deduced that b_o/s is a constant for fixed values of θ_r and θ_o. When a/s increases, the conical shell becomes longer for constant θ_r and θ_o. We shall observe from the mode shape figures that the S-1 mode for a CFFF shell is generally a spanwise bending mode which has a lower frequency (smaller λ') if it is longer (increasing a/s). This fact is

Table 7. Frequency parameter $\lambda' = \omega b_o \sqrt{(\rho/E)}$ for a CCCC shallow conical shell with $\nu = 0.3$ and $\theta_o = 15$

			Mode sequence number							
θ_r				Symmetric modes				Antisymmetric modes		
(deg)	a/s	a/h	S-1	S-2	S-3	S-4	A-1	A-2	A-3	A-4
15	0.10	1000	0.10681	0.11541	0.12682	0.14232	0.071195	0.095980	0.12279	0.14701
		60	0.43895	0.47820	0.54211	0.64181	0.95563	1.0222	1.0972	1.2001
		30	0.73618	0.81332	0.93996	1.1317	1.7138	1.8224	1.9452	2.1151
	0.15	1000	0.15409	0.16404	0.17264	0.18065	0.096718	0.10945	0.12409	0.14119
		60	0.58890	0.63409	0.67691	0.73058	1.3650	1.4522	1.5272	1.5977
		30	0.99366	1.0690	1.1416	1.2363	2.2390	2.3639	2.4710	2.5730
	0.20	1000	0.19272	0.20330	0.21215	0.22006	0.12597	0.13909	0.15032	0.16099
		60	0.73536	0.79262	0.84343	0.89150	1.7139	1.8183	1.9080	1.9902
		30	1.1982	1.2843	1.3603	1.4326	2.5855	2.7206	2.8366	2.9421
30	0.10	1000	0.067456	0.099372	0.13464	0.14054	0.076505	0.10350	0.11593	0.13191
		60	0.33352	0.42011	0.58737	0.83457	0.52699	0.64671	0.83145	1.0838
		30	0.49431	0.69614	1.0494	1.5372	1.0103	1.2213	1.5583	2.0213
	0.15	1000	0.086993	0.10221	0.12393	0.14974	0.071317	0.10806	0.13597	0.15380
		60	0.39470	0.44989	0.54576	0.69413	0.76420	0.85515	0.96630	1.1190
		30	0.63145	0.74856	0.94823	1.2414	1.4241	1.5806	1.7713	2.0352
	0.20	1000	0.11279	0.12639	0.13975	0.15550	0.077281	0.10403	0.13245	0.15836
		60	0.47012	0.52542	0.59122	0.68894	0.99833	1.1063	1.2040	1.3101
		30	0.78423	0.88781	1.0158	1.2041	1.7856	1.9609	2.1184	2.2909
45	0.10	1000	0.076327	0.10590	0.11606	0.15352	0.073779	0.11502	0.12235	0.15037
		60	0.33845	0.48971	0.69350	0.76990	0.40496	0.60895	0.91052	1.0768
		30	0.47239	0.82781	1.3310	1.4124	0.76299	1.1361	1.7143	2.0717
	0.15	1000	0.071399	0.10725	0.14355	0.14479	0.081636	0.10931	0.12386	0.14078
		60	0.36038	0.44785	0.61639	0.86781	0.53969	0.66898	0.86015	1.1198
		30	0.52009	0.72608	1.0869	1.5872	1.0324	1.2592	1.6067	2.0829
	0.20	1000	0.082915	0.10509	0.13474	0.16672	0.078902	0.11663	0.14332	0.14629
		60	0.40143	0.46741	0.58087	0.75601	0.70323	0.81583	0.95113	1.1339
		30	0.61426	0.75790	0.99997	1.3505	1.3242	1.5208	1.7560	2.0768
60	0.10	1000	0.084607	0.11730	0.13734	0.14668	0.085983	0.10878	0.13669	0.16091
		60	0.38008	0.57170	0.60724	0.84286	0.38728	0.67716	0.83465	1.0905
		30	0.52810	1.0457	1.0830	1.6061	0.68962	1.2451	1.6223	2.0695
	0.15	1000	0.081319	0.11645	0.12439	0.16528	0.079817	0.12162	0.13508	0.16448
		60	0.37887	0.50885	0.75588	0.77112	0.44150	0.63380	0.91147	1.1963
		30	0.50998	0.82247	1.3523	1.4702	0.83096	1.1728	1.6995	2.2859
	0.20	1000	0.077156	0.11889	0.14619	0.15830	0.087690	0.11760	0.13623	0.15400
		60	0.40004	0.48953	0.66001	0.91758	0.54859	0.69263	0.89455	1.1658
		30	0.55473	0.76736	1.1401	1.6594	1.0453	1.2958	1.6613	2.1593

generally true in Tables 5 and 6. The effects of a/s on the other higher modes are more complicated because they could be a mixture of chordwise and spanwise bending modes.

From the tables, it can be observed that λ' for the CCCC shell is higher than the CFFF shell for a fixed shell configuration. The CCCC shell has more boundary constraints which render higher structural stiffness and thus a higher vibration frequency.

A set of vibration modes in mid-surface contour and three-dimensional displacement plots is presented in Figs 2 and 3 for the CFFF and CCCC shells, respectively. The a/s ratio varies from 0.1 to 0.2 while the thickness ratio h/a is 1/10 ($h/b_o = 0.072$ for $a/s = 0.1$ and $h/b_o = 0.145$ for $a/s = 0.2$). It is observed that the S-1 modes are the first spanwise bending modes whereas the A-1 modes are the first torsional modes for the CFFF shell in Fig. 2. The frequency for the S-1 mode decreases when the shell becomes longer as a/s steps up from 0.1 to 0.2. The in-plane dominant vibration modes are the A-2 modes where $\lambda' = 1.0641$ and 0.39430 for the CFFF shell. For the vibration modes of the CCCC shell as illustrated in Fig. 3, all the boundary surfaces are undeformed because they are fully clamped.

5. CONCLUSIONS

This paper studies the free vibration of moderately thick shallow conical shells. The energy integrals incorporating the thickness-shear and rotary inertia effects are approximated by a refined first-order shear deformation theory. The Lamé parameter for the

Table 8. Frequency parameter $\lambda' = \omega h_0\sqrt{(\rho/E)}$ for a CCCC shallow conical shell with $\nu = 0.3$ and $\theta_o = 30°$

			Mode sequence number							
θ				Symmetric modes				Antisymmetric modes		
(deg)	a/s	a/h	S-1	S-2	S-3	S-4	A-1	A-2	A-3	A-4
15	0.10	1000	0.088404	0.13758	0.15133	0.17783	0.094988	0.13069	0.16447	0.17701
		30	0.64435	0.80791	1.1187	1.5692	1.0351	1.2530	1.5864	2.0352
		15	0.94218	1.2871	1.8719	2.6433	1.8293	2.1696	2.6984	3.3977
	0.15	1000	0.096115	0.12592	0.16643	0.20683	0.10185	0.14095	0.16901	0.17227
		30	0.76503	0.86657	1.0394	1.3036	1.4493	1.6081	1.7996	2.0597
		15	1.1580	1.3444	1.6569	2.1030	2.3442	2.5681	2.8463	3.2245
	0.20	1000	0.12157	0.14033	0.16401	0.19398	0.098061	0.14731	0.18289	0.21121
		30	0.90211	1.0013	1.1158	1.2834	1.8099	1.9856	2.1432	2.3147
		15	1.3588	1.5108	1.6988	1.9728	2.6872	2.9068	3.1091	3.3440
30	0.10	1000	0.10386	0.13892	0.16858	0.18938	0.10633	0.13373	0.16729	0.18130
		30	0.64929	1.0702	1.1535	1.6253	0.73552	1.2500	1.7238	2.0007
		15	0.93442	1.8310	2.1232	2.9476	1.3237	2.2379	3.1550	3.5565
	0.15	1000	0.10148	0.14001	0.14886	0.20253	0.095719	0.15100	0.15637	0.19290
		30	0.65265	0.89619	1.3455	1.5749	0.88694	1.2118	1.6915	2.3279
		15	0.92246	1.4506	2.3000	2.7837	1.6083	2.1404	2.9400	3.8683
	0.20	1000	0.094945	0.14656	0.16518	0.19022	0.10273	0.14081	0.17674	0.18923
		30	0.70339	0.87075	1.1780	1.6281	1.1132	1.3562	1.6923	2.1428
		15	1.0201	1.3676	1.9493	2.7231	1.9547	2.3324	2.8597	3.5564
45	0.10	1000	0.12581	0.14916	0.17764	0.20978	0.12698	0.14489	0.19028	0.21097
		30	0.77161	1.0014	1.4777	1.7772	0.78794	1.3401	1.5849	2.1179
		15	1.1720	1.8354	2.6256	3.2742	1.3656	2.5208	2.8678	3.8995
	0.15	1000	0.11159	0.14595	0.18028	0.20314	0.11347	0.14398	0.17978	0.18904
		30	0.70195	1.1233	1.1937	1.6782	0.76675	1.2981	1.7702	2.0637
		15	0.98676	1.8953	2.1858	3.0349	1.3669	2.3081	3.2372	3.6564
	0.20	1000	0.10860	0.15470	0.16508	0.20225	0.10572	0.15180	0.16800	0.20986
		30	0.70449	0.99108	1.4868	1.5146	0.86353	1.2551	1.8175	2.2708
		15	0.97049	1.5979	2.5921	2.6647	1.5629	2.2144	3.1688	3.9234
60	0.10	1000	0.15412	0.17408	0.20929	0.26886	0.15563	0.17489	0.21531	0.27330
		30	0.95121	1.0570	1.5682	1.9635	0.94563	1.2560	1.9656	2.0391
		15	1.5050	1.8952	2.9702	3.5513	1.6127	2.3512	3.6957	3.7485
	0.15	1000	0.13270	0.15896	0.19124	0.21945	0.13349	0.15496	0.20659	0.22018
		30	0.81412	1.0669	1.4381	1.7790	0.81142	1.4481	1.5535	2.1593
		15	1.1734	1.9327	2.4830	3.2374	1.3822	2.6874	2.7867	3.9492
	0.20	1000	0.12310	0.15515	0.19728	0.21804	0.12348	0.15871	0.19791	0.20042
		30	0.78069	1.1992	1.2416	1.7402	0.80699	1.3647	1.8060	2.1509
		15	1.0644	1.9901	2.2435	3.1334	1.4149	2.3993	3.3017	3.7926

transverse shear strain component is considered, which results in the replacement of a term in transverse shear strains which varies linearly through the shell thickness in contrast to existing constant distributions. The energy functional is minimized in accordance with the Ritz extremum energy procedure. A computational efficient numerical method with orthogonally generated global shape functions has been developed which requires no domain discretization or mesh generation. The flexibility of the method in accounting for various boundary conditions is demonstrated through two numerical examples: the cantilever and the fully clamped conical shells.

A convergence study has been presented to show the downward convergence of eigenvalues and to determine the number of terms required for satisfactory convergent results. The solutions have been compared with thin conical shell solutions by assigning a small thickness ratio to the presented formulation. Excellent agreement has been obtained. The effects of the missing Lamé parameter for the transverse shear strain have been shown to influence the vibration frequencies significantly. A set of benchmark frequency parameters has been presented for a wide class of shell configurations for both cantilever and fully clamped shells. It has been found that an increase in thickness results in a higher frequency for a transverse dominant bending mode. For the cantilever shell, a decline in frequency for the first spanwise bending mode was observed when the shell became longer. These facts have been reflected through some selected three-dimensional displacement vibration modes shapes.

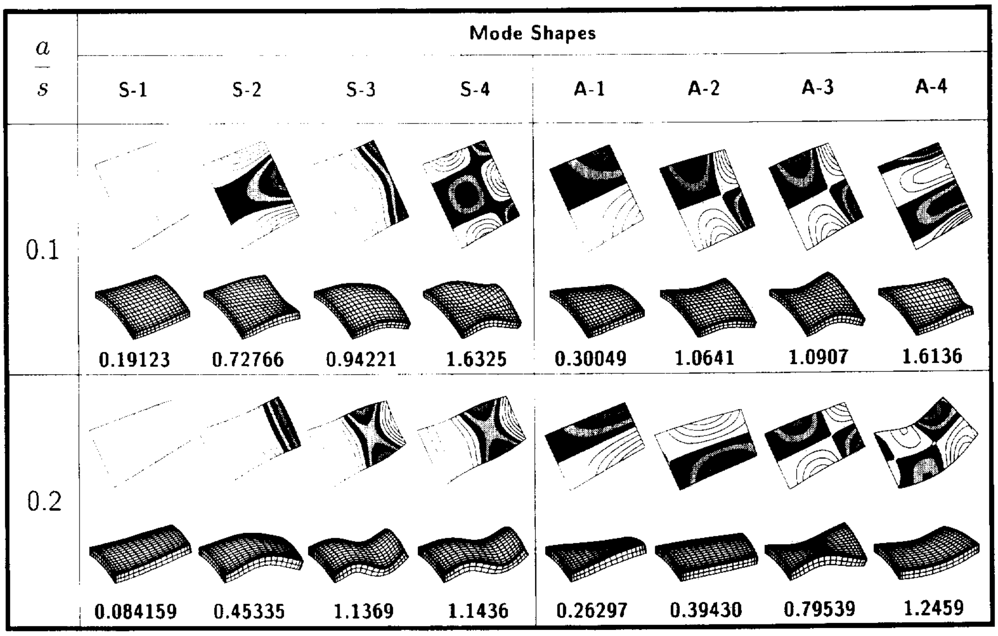

Fig. 2. Mid-surface contour and three-dimensional vibration mode shapes of a CFFF shallow conical shell with $v = 0.3$, $\theta_o = 30^\circ$, $\theta_v = 30^\circ$ and $h/a = 0.1$.

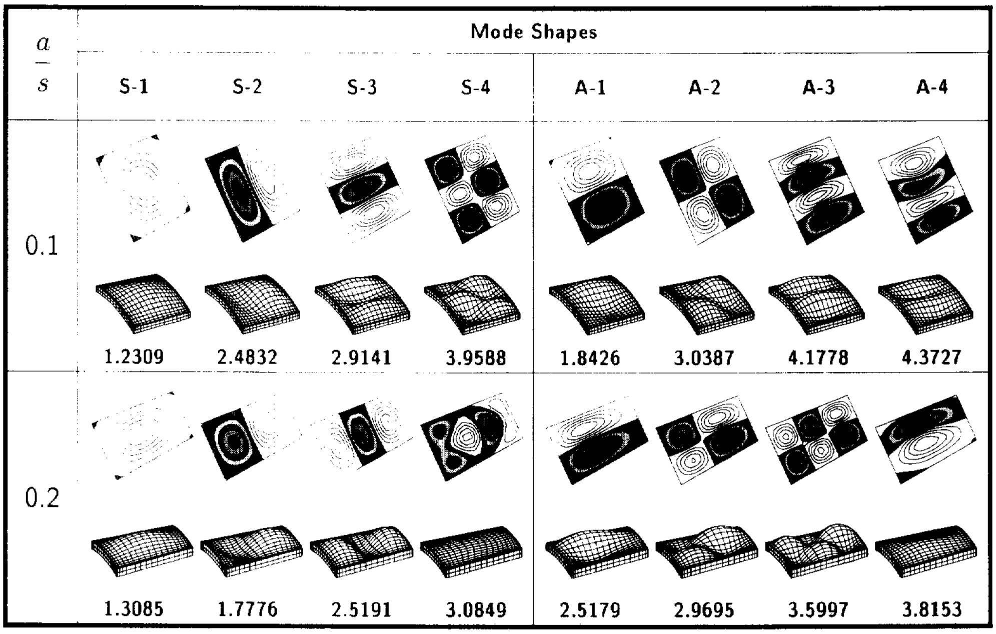

Fig. 3. Mid-surface contour and three-dimensional vibration mode shapes of a CCCC shallow conical shell with $v = 0.3$, $\theta_o = 30^\circ$, $\theta_v = 30^\circ$ and $h/a = 0.1$.

REFERENCES

Chang, C. H. (1981). Vibration of conical shells. *Shock Vibr. Digest* **13,** 9–17.

Cheung, Y. K., Li, W. Y. and Tham, L. G. (1989). Free vibration analysis of singly curved shell by spline finite strip method. *J. Sound Vibr.* **128,** 411–422.

Irie, T., Yamada, G. and Kaneko, Y. (1982). Free vibration of a conical shell with variable thickness. *J. Sound Vibr.* **82,** 83–94.

Irie, T., Yamada, G. and Tanaka, K. (1984). Natural frequencies of truncated conical shells. *J. Sound Vibr.* **92,** 447–453.

Kabir, H. R. H. and Chaudhuri, R. A. (1991). Free vibration of shear-flexible anti-symmetric single-ply doubly curved panels. *Int. J. Solids Structures* **28,** 17–32.

Leissa, A. W. (1981). Vibrational aspects of rotating turbomachinery blades. *Appl. Mech. Rev.* **34,** 629–635.

Leissa, A. W., Lee, J. K. and Wang, A. J. (1982). Rotating blade vibration analysis using shells. *ASME J. Engng Pwr* **104,** 296–302.

Liew, K. M. and Lim, C. W. (1994). Vibratory characteristics of cantilevered rectangular shallow shells of variable thickness. *AIAA Journal* **32,** 387–396.

Liew, K. M., Lim, C. W. and Ong, L. S. (1994). Vibration of pretwisted cantilevered shallow conical shells. *Int. J. Solids Structures* **31,** 2463–2476.

Liew, K. M., Lim, M. K., Lim, C. W., Li, D. B. and Zhang, Y. R. (1995). Effects of initial twist and thickness variation on the vibration behaviour of shallow conical shells. *J. Sound Vibr.* **180,** 271–296.

Lim, C. W. and Liew, K. M. (1995). The vibration behaviour of shallow conical shells by a global Ritz formulation. *Engng Structures* **17,** 63–70.

Mindlin, R. D. (1951). Influence of rotatory inertia and shear on flexural motion of isotropic, elastic plates. *ASME J. Appl. Mech.* **18,** 31–38.

Rao, J. S. (1973). Natural frequencies of turbine blading—a survey. *Shock Vibr. Digest* **5,** 3–16.

Rao, J. S. (1977). Turbine blading excitation and vibration. *Shock Vibr. Digest* **9,** 15–22.

Rao, J. S. (1980). Turbomachine blade vibration. *Shock Vibr. Digest* **12,** 19–26.

Reddy, J. N. (1984). Exact solutions of moderately thick laminated shells. *ASCE J. Engng Mech.* **110,** 794–809.

Sivadas, K. R. and Ganesan, N. (1992). Vibration analysis of thick composite clamped conical shells of varying thickness. *J. Sound Vibr.* **152,** 27–37.

Srinivasan, R. S. and Hosur, V. (1989). Axisymmetric vibration of thick conical shells. *J. Sound Vibr.* **135,** 171–176.

Srinivasan, R. S. and Krishnan, P. A. (1987). Free vibration of conical shell panels. *J. Sound Vibr.* **117,** 153–160.

Takahashi, S., Suzuki, K., Anzai, E. and Kosawada, T. (1982). Axisymmetric vibration of conical shells with variable thickness. *Bull. Jap. Soc. Mech. Engrs* **25,** 1771–1780.

Takahashi, S., Suzuki, K. and Kosawada, T. (1985). Vibrations of conical shells with variable thickness (continued). *Bull. Jap. Soc. Mech. Engrs* **28,** 117–123.

Takahashi, S., Suzuki, K. and Kosawada, T. (1986). Vibrations of conical shells with variable thickness (third report: analysis by the higher order improved theory). *Bull. Jap. Soc. Mech. Engrs* **29,** 4306–4311.

Teichmann, D. (1985). An approximation of the lowest eigen frequencies and buckling loads of cylindrical and conical shell panels under initial stress. *AIAA J.* **23,** 1634–1637.